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While this article does not give an exhaustive description of the construction and uses of the astrolabe, it is hoped that it may at least serve to show by what means the sciences of astronomy and surveying attained so great a degree of development before the days of Tycho Brahe and Galileo.

A USEFUL PRINCIPLE IN CURVE TRACING.

By ARNOLD EMCH, University of Illinois.

1. The usual elementary methods of curve tracing in rectangular coördinates consist:

(1) In plotting points of the curve by assigning arbitrary values to one of the variables and finding the corresponding values of the other variable;
(2) In determining the tangents at these points by means of the derivatives;
(3) In finding the intersections of the curve with definite straight lines, or other conveniently chosen curves;
(4) In establishing possible properties of symmetry;
(5) In determining concavity and convexity, maxima and minima;
(6) In determining the asymptotes;
(7) In determining possible singular and inflexional points.

This list of tests and different steps to be taken in the investigation of a curve may, of course, be extended according to the difficulties and the nature of the problem.[1]

In ordinary curve tracing little use is made of other algebraic methods than those mentioned above. It is the purpose of this note to show the effectiveness of a certain algebra-geometric method in elementary curve tracing.

2. Restricting ourselves to algebraic curves, let such a curve be represented by the equation

$$F(x, y) = 0 \tag{1}$$

in which $F(x, y)$ is an irreducible polynomial. It is always possible to write $F(x, y) = 0$ in the identical form

$$F(x, y) \equiv \phi_1(x, y)\psi_2(x, y) - \phi_2(x, y)\psi_1(x, y) = 0, \tag{2}$$

where ϕ_1, ϕ_2, ψ_1, ψ_2 are also polynomials in x and y, of which some may reduce to constants. If it should prove to be convenient, we might use

$$F(x, y) - G(x, y) + G(x, y)$$

in place of $F(x, y)$. Thus, as $G(x, y)$ may be any polynomial, we may resolve $F(x, y)$ in an infinite number of ways into the form (2). Equation (2) is evidently the result of the elimination of λ between the simultaneous equations

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[1] See Percival Frost: *An Elementary Treatise on Curve Tracing*, pp. 177–187 (1911).

$$\phi_1(x, y) + \lambda\phi_2(x, y) = 0, \tag{3}$$

$$\psi_1(x, y) + \lambda\psi_2(x, y) = 0. \tag{4}$$

All sets of points (x, y) that satisfy (3) and (4) simultaneously also satisfy (2). Conversely, any set (x_1, y_1) that satisfies (2), with the possible exception of a limited number of sets that satisfy (2) for some particular value of λ, is a simultaneous solution of (3) and (4). As a matter of fact

$$\frac{\phi_1(x_1, y_1)}{\phi_2(x_1, y_1)} = \frac{\psi_1(x_1, y_1)}{\psi_2(x_1, y_1)} = -\lambda_1,$$

for all sets (x_1, y_1) of (2), for which these ratios exist. For a variable parameter λ, (3) and (4) represent two projective pencils of curves, whose product is the given curve (2). Hence the well-known

THEOREM. *An algebraic curve may always be generated by two projective pencils of algebraic curves.*

This theorem forms the basis of Steiner's method of investigation of algebraic curves and is closely connected with Noether's famous theorem concerning the possibility of representing $F(x, y)$ in the form given in (2), when ψ_1 and ψ_2 are given.[1]

3. The possibility of representing a polynomial in the form given in (2) and by Steiner's theorem may be utilized with great advantage in cases where the explicit representation of one variable in terms of the other involves the solution of an equation of higher than the second or third degree. But the method may also be used in any other case and may be embodied in the following principle:

Resolve $F(x, y)$ *into the identical form* $\phi_1(x, y)\psi_2(x, y) - \phi_2(x, y)\psi_1(x, y)$, *in such a manner that the equations*

$$\phi_1(x, y) + \lambda\phi_2(x, y) = 0,$$

$$\psi_1(x, y) + \lambda\psi_2(x, y) = 0,$$

assume as simple forms as possible (geometrically). Construct these curves for as many values of λ *as seem necessary; then for every value of* λ *the two curves intersect in points of the given curve.*

4. Examples:

Conics. The application of the principle to equations of the second degree leads to the well-known projective theory of conics. ϕ_1, ϕ_2, ψ_1, ψ_2 are linear in x and y, and (3) and (4) become ordinary projective pencils of straight lines.

A quartic: $x^4 - y^4 - x^2y^2 + x^3y - x^3 + y^2 = 0$. This may be written in the form

$$x^3(x + y - 1) - y^2(x^2 + y^2 - 1) = 0, \tag{5}$$

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[1] See references to this subject in Pascal's *Repertorium der höheren Mathematik*, Vol. II, first part, 2d ed. (1910), pp. 287–289 and pp. 306–307.

so that the quartic may be generated by the two projective pencils,

$$x^2 + y^2 - 1 - 2\lambda(x + y - 1) = 0, \tag{6}$$

$$-x^3 + 2\lambda y^2 = 0. \tag{7}$$

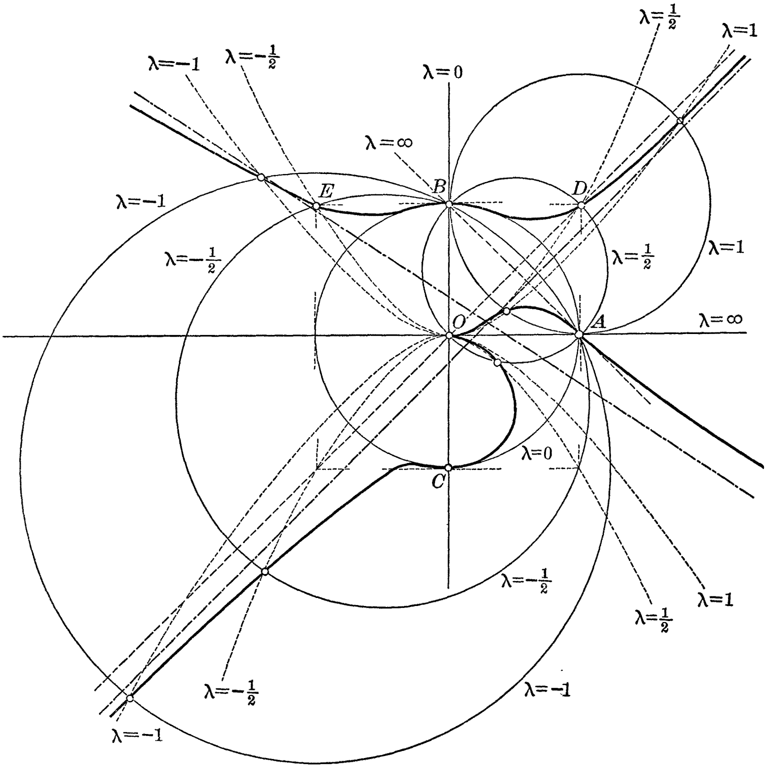
Equation (6) may be written in the form

$$(x - \lambda)^2 + (y - \lambda)^2 = (\lambda - 1)^2 + \lambda^2$$

and represents a circle whose center has the coördinates (λ, λ), and whose radius is $\sqrt{(\lambda - 1)^2 + \lambda^2}$, and which passes through the points $A(1, 0)$ and $B(0, 1)$ (see figure.) Hence, (6) represents a pencil of circles through A and B, with their centers on the bisector of the first and third quadrants. Writing (7) in the form

$$y^2 = \frac{x^3}{2\lambda}, \tag{9}$$

a semi-cubical parabola is obtained, which passes through the points $(2\lambda, \pm 2\lambda)$, which has a cusp at the origin with the x-axis as a tangent, and which may easily



be plotted for a given value of λ. In fact, when μ is any real number, all points $(2\lambda\mu^2, \pm 2\lambda\mu^3)$ lie on (9). For $\lambda > 0$ the cubics lie on the right side of the y-axis;

for $\lambda < 0$ on the left side. For $\lambda = \frac{1}{2}$ the circle (6) passes through the origin and cuts the corresponding cubic $y^2 = x^3$ in two coincident points. Hence, as for this value of λ the circle is not tangent to the cubic at the origin, the quartic must have a singular point at the origin. For $\lambda = \frac{1}{2} - \epsilon$ (ϵ = arbitrarily small positive quantity) the circle does not cut the cubic in real points in the neighborhood of the origin. For $\lambda = \frac{1}{2} + \epsilon$ there are two real points of intersection arbitrarily close to the origin on the right side of the y-axis, one below, the other above the x-axis. From this we conclude that the singularity of the quartic at the origin is a cusp. When $\lambda \doteq 0$, (6) and (7) approach $x^2 + y^2 - 1 = 0$ and $x^3 = 0$, which shows that at B and C (see figure) the circle has three points in common with the quartic. In other words, the unit-circle osculates the quartic at B and C. But at B we have an additional point of intersection of the unit-circle with the line $x + y - 1 = 0$, so that at B the unit-circle has a contact of the third degree with the quartic.

When $\lambda \doteq \infty$, (6) and (7) approach $x + y - 1 = 0$ and $y^2 = 0$, which shows that the quartic touches the line $x + y - 1 = 0$ at A. But another point of intersection, that of the same line with $x^2 + y^2 - 1 = 0$, at A, together with the other two, makes $x + y - 1 = 0$ an inflexional tangent at A. It is easily seen that the quartic also passes through $D(1, 1)$ and $E(-1, 1)$. Any number of other points of the quartic may be obtained as intersections of corresponding curves of the two projective pencils (6) and (7). In the figure, such intersections are shown for $\lambda = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \infty$. The two real asymptotes were determined by the method of substitution.

Lemniscate.[1] The equation of this bicircular rational quartic is

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0, \tag{10}$$

and may be written in the form

$$(x^2 + y^2)(x^2 + y^2) - 2a^2(x + y)(x - y) = 0. \tag{11}$$

It is the product of the two projective pencils of circles

$$(x^2 + y^2) + \lambda 2a(x + y) = 0, \tag{12}$$

$$a(x - y) + \lambda(x^2 + y^2) = 0. \tag{13}$$

These equations may be written in the form

$$(x + \lambda a)^2 + (y + \lambda a)^2 = 2\lambda^2 a^2, \tag{14}$$

$$\left(x + \frac{a}{2\lambda}\right)^2 + \left(y - \frac{a}{2\lambda}\right)^2 = \frac{a^2}{2\lambda^2}. \tag{15}$$

They represent two projective pencils of circles which all pass through the origin. All circles of (14) are tangent to the line $x - y = 0$; all circles of (15) are tangent to the line $x + y = 0$. The centers of the circles (14) lie on $x + y = 0$, those of (15) on $x - y = 0$. The construction of the circles (14) and (15) for a given λ is therefore extremely simple, and consequently also the construction of points of the lemniscate as intersections of (14) and (15). It is obvious that the circular points at infinity (isotropic points) and the origin are singular points (inflexional double points) of the quartic.

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[1] The name of this curve is derived from the Greek λημνίσχος, which means a loop in shape of the figure 8, and dates back to Jac. Bernoulli who discovered this curve. See his works (Genevae, 1744), Vol. 1, p. 609.

In an equally simple manner the theorem can be established, by the same principles, that all bicircular quartics may be generated by projective pencils of circles.

It is hoped, however, that the few examples explained above will suffice to show the great value of the principle for actual constructive purposes.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

A Short Course in Elementary Mechanics for Engineers. By CLIFFORD NEWTON MILLS, B.S., A.M., Assistant Professor of Mathematics, South Dakota State College of Agriculture and Mechanic Arts. The D. Van Nostrand Company, New York, 1916. xi + 127 pages. $1.00.

This little book contains in a condensed form a discussion of the simpler subjects in kinematics, kinetics and statics. As only uniform motion, uniformly accelerated motion, and the forces which produce such motions are considered, the author has willed to treat them without the use of mathematics beyond trigonometry. The course is written from the definition viewpoint, much detailed discussion being omitted. For this reason and for the further reason that the clean-cut demonstrations of the calculus are not used the reasoning may not be always as rigid as the old-time mathematicians would like.

The engineering unit, the pound, is used throughout as the unit of force; it is defined as "that force which will accelerate a mass of one pound g feet per second per second." The pound of mass is not definitely defined but these relations are given,

$$\text{Weight} = \text{mass} \times g$$

$$\text{Mass} = W/g$$

thus avoiding hair-splitting distinctions.

Problems involving center of gravity and moment of inertia have been ingeniously worked out without the calculus; true, the summation sign, Σ, has been employed, but the summations are evaluated by algebraic processes.

Friction which might have been treated in an elementary manner, and which is of great importance to machinists and others who would have use for this book, seems to have been omitted. There is a large supply of well-graded problems.

The reviewer does not wish to give his full endorsement to a work in mechanics